6th International Conference on Advances in Experimental Structural Engineering
11th International Workshop on Advanced Smart Materials and Smart Structures Technology
August 1-2, 2015, University of Illinois, Urbana-Champaign, United States

Experimental Studies of Two-Stage Covariance-Based Multi-Sensing Damage Detection Method

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ABSTRACT

Local damages in a large structure are difficult to be accurately identified by using a limited number of single-type sensors. Multi-sensing damage detection methods have thus been put forward. However, distinct properties and limited capacities of different sensors considerably complicate and impede the development of successful multi-sensing structural damage detection methods. In this regard, a new multi-sensing damage detection method has recently been proposed by the authors, which includes a new covariance-based multi-sensing (CBMS) damage detection index and a two-stage damage detection procedure. To examine the feasibility and accuracy of the proposed method, experimental investigation was carried out. A simply supported overhanging steel beam of a total of length of 4 m was built in a structural laboratory with a roller support and a pin support. Two types of sensors including accelerometers and strain gauges are used to measure the structural responses. The single damage scenario was considered. The experimental results show that the two-stage CBMS damage detection method is more sensitive to local structure defects, and that identification results are more accurate through integrated use of multi-type sensor data. The feasibility and efficiency of the proposed multi-sensing damage detection method are validated. .

KEYWORDS: *Damage Detection, Multi-Sensing, Cross-Covariance, Two-Stage Approach*

INTRODUCTION

An important issue in the application of structural health monitoring (SHM) is structural damage detection which has been extensively investigated. Accelerometers are relatively reliable sensors for damage detection which with acceleration responses usually contain global information of a structure. However, with the understanding that damage is a local phenomenon, using only accelerometers has its limitations for damage detection, meanwhile the SHM system usually contains multiple types of sensors. Therefore the multi-sensing approaches for damage detection have a great potential for improving the damage detection result. Studer and Peters (2004) presented a strategy using multi-metric data of strain, integrated strains and gradients measured from optical fiber sensors for damage identification. Law et al. (2005) used a wavelet-based approach to combine acceleration and strain response for damage identification and achieved better damage detection results than using the two measurements separately. Chan et al. (2006) proposed an integrated GPS-accelerometer data processing technique for improving the accuracy of measurement data. Zhang et al. (2011) suggested an integrated optimal sensor placement of displacement transducer and strain gauges for better response reconstruction. Sim et al. (2011) presented a flexibility-based method combining acceleration and strain responses for structural damage detection. More recently, Lee et al. (2013) developed a modified GDM (global-deviation method) which can be effectively utilized in detecting damage based on the mixed measurements of accelerometers and strain gages. Sung et al.(2014) found that the damage metric estimated from acceleration measurement is insensitive to damage near the hinged support of a bridge, and therefore they proposed a multi-scale sensing and diagnosis system for bridge health monitoring based on a two-step improvement approach using accelerometers and gyroscopes. However, there is still few literatures found for detail discussion of a standard and unified framework for multi-sensing structural damage detection and condition assessment. This paper aims at developing a new framework to address this problem in the time domain. Cross-covariance functions are used to assimilate heterogeneous data from multi-types of sensors and put various structural responses together. The cross-covariance functions have the merit in reducing the adverse impact of random measurement noise in structural responses. A cross-covariance matrix is then formed and the covariance-based multi-sensing (CBMS) damage detection vector is defined as an integrated damage index. The sensitivity-based method is used to derive the formulation for the CBMS damage detection method in terms of the CBMS vector. The CBMS method is then proposed in two stages for detecting damage location and severity consecutively. Experimental study is finally performed to investigate the feasibility and accuracy of the proposed framework using an overhanging beam with multiple damage scenarios.

THEORETICAL FRAMEWORK

Data Set of Normalized Multi-sensing Response

Multi-sensing information from multi-type sensors in a structural health monitoring system installed in a structure often includes acceleration, displacement and strain responses. The recorded data from each channel are assumed with the same sampling rate and length in this study, otherwise a data pre-processing scheme is necessary before the subsequent study of a new damage index. In consideration that acceleration, displacement and strain responses are of different units and magnitudes, a normalized multi-sensing data set $[\mathbf{Y}(t)]=[\frac{\mathbf{Y}^a(t)}{\sigma^a},\frac{\mathbf{Y}^d(t)}{\sigma^d},\frac{\mathbf{Y}^\varepsilon(t)}{\sigma^\varepsilon}]^{\mathrm{T}}$ is formed for all the selected multi-sensing measurement responses. Here, $\mathbf{Y}^a(t)$, $\mathbf{Y}^d(t)$ and $\mathbf{Y}^\varepsilon(t)$ are the time history records of all acceleration, displacement and strain responses, respectively, while σ^a, σ^d and σ^ε are the average standard deviation constants which are computed as the mean of the standard deviations of all acceleration, displacement and strain response records, respectively. Finally the normalized multi-sensing data set is produced as:

$$[\mathbf{Y}(t)]=\begin{bmatrix} y_1^a(t_1) & y_1^a(t_2) & \cdots & y_1^a(t_j) & \cdots & y_1^a(t_{mt}) \\ y_2^a(t_1) & y_2^a(t_2) & \cdots & y_2^a(t_j) & \cdots & y_2^a(t_{mt}) \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ y_{m_a+1}^d(t_1) & y_{m_a+1}^d(t_2) & \cdots & y_{m_a+1}^d(t_j) & \cdots & y_{m_a+1}^d(t_{mt}) \\ y_{m_a+2}^d(t_1) & y_{m_a+2}^d(t_2) & \cdots & y_{m_a+2}^d(t_j) & \cdots & y_{m_a+2}^d(t_{mt}) \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ y_{m_a+m_d+1}^\varepsilon(t_1) & y_{m_a+m_d+1}^\varepsilon(t_2) & \cdots & y_{m_a+m_d+1}^\varepsilon(t_j) & \cdots & y_{m_a+m_d+1}^\varepsilon(t_{mt}) \\ y_{m_a+m_d+2}^\varepsilon(t_1) & y_{m_a+m_d+2}^\varepsilon(t_2) & \cdots & y_{m_a+m_d+2}^\varepsilon(t_j) & \cdots & y_{m_a+m_d+2}^\varepsilon(t_{mt}) \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \end{bmatrix} \qquad (1)$$

where $y_i^a(t_j)$ is the i^{th} location's normalized acceleration response time history at the time $t_j\left(i=1,2,\ldots,\ m_a;\ j=1,2,\ldots mt\right)$; $y_{m_a+i}^d(t_j)$ is the $(m_a+i)^{th}$ location's normalized displacement response time history at the time $t_j\left(i=1,2,\ldots,\ m_d;\ j=1,2,\ldots mt\right)$; $y_{m_a+m_b+i}^\varepsilon(t_j)$ is the $(m_a+m_b+i)^{th}$ location's normalized strain response time history at the time $t_j\left(i=1,2,\ldots,\ m_s;\ j=1,2,\ldots mt\right)$; mt is the total number of sampling data points in one record; m_a, m_d and m_s denote the total number of accelerometers, displacement transducers and strain gauges respectively; the total number of measurement locations is $ms=m_a+\ m_d+\ m_s$.

Covariance Function of Multi-Sensing Responses

To fully utilize multi-sensing information and significantly reduce the adverse impact of measurement noise, the cross-covariance function of the two normalized response time histories (Bendat and Piersol 1993), $x_p(t)$ and $x_l(t)$, is considered in this study.

$$C_{pl}(\tau)=\mathrm{E}[\{y_p(t)-\mu_p\}\cdot\{y_l(t+\tau)-\mu_l\}] \qquad (2)$$

where $\mathrm{E}[\bullet]$ is the expectation operation; n is the total data number used for covariance computation; μ_p and μ_l are the mean values of the normalized structural responses at the location p and l respectively; and τ is the time lag; $v_p(t)$ and $v_l(t+\tau)$ are the measurement noise at the location p and l, respectively. The measurement noise is assumed to be a white noise Gaussian process with $\mathrm{E}(v_p) = \mathrm{E}(v_l) = 0$. When $\mu_p = \mu_l = 0$, the cross-covariance function $C_{pl}(\tau)$ becomes the cross-correlation function, and furthermore when $p = l$ it becomes the auto-correlation function.

Covariance-Based Multi-Sensing (CBMS) Damage Detection Vector

The cross-covariance matrix of multi-sensing responses is the function of time lag τ and total number of time lag is assumed as nt, then the cross-covariance matrix can be given as follows:

$$\mathbf{C}_{pl} = \begin{bmatrix} C_{p_1 l_1}(\tau_1) & C_{p_1 l_1}(\tau_2) & \cdots & C_{p_1 l_1}(\tau_{kt}) & \cdots & C_{p_1 l_1}(\tau_{nt}) \\ C_{p_1 l_2}(\tau_1) & C_{p_1 l_2}(\tau_2) & \cdots & C_{p_1 l_2}(\tau_{kt}) & \cdots & C_{p_1 l_2}(\tau_{nt}) \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ C_{p_1 l_{ms}}(\tau_1) & C_{p_1 l_{ms}}(\tau_2) & \cdots & C_{p_1 l_{ms}}(\tau_{kt}) & \cdots & C_{p_1 l_{ms}}(\tau_{nt}) \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ C_{p_q l_q}(\tau_1) & C_{p_q l_q}(\tau_2) & \cdots & C_{p_q l_q}(\tau_{kt}) & \cdots & C_{p_q l_q}(\tau_{nt}) \\ C_{p_q l_{q+1}}(\tau_1) & C_{p_q l_{q+1}}(\tau_2) & \cdots & C_{p_q l_{q+1}}(\tau_{kt}) & \cdots & C_{p_q l_{q+1}}(\tau_{nt}) \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ C_{p_q l_{ms}}(\tau_1) & C_{p_q l_{ms}}(\tau_2) & \cdots & C_{p_q l_{ms}}(\tau_{kt}) & \cdots & C_{p_q l_{ms}}(\tau_{nt}) \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ C_{p_{ms} l_{ms}}(\tau_1) & C_{p_{ms} l_{ms}}(\tau_2) & \cdots & C_{p_{ms} l_{ms}}(\tau_{kt}) & \cdots & C_{p_{ms} l_{ms}}(\tau_{nt}) \end{bmatrix} \in \mathbb{R}^{ms \times nt} \tag{3}$$

where the subscript q and kt denote the q^{th} measurement location and the kt^{th} time lag with $\tau_1 < \tau_2 < \cdots < \tau_{nt}$. Since the cross-covariance function in Eq.(2) is a decay function and only the first nt time lags are selected in Eq.(3) for the subsequent study. Since the cross-covariance matrix contains multi-sensing information from multi-type sensors, the covariance-based multi-sensing (CBMS) damage detection vector $\mathbf{V}_{pl}$ is considered in this study as an integrated damage index vector by drawing the components of the matrix $\mathbf{C}_{pl}$ column by column in Eq. (3) as follows:

$$\begin{aligned} \mathbf{V}_{pl} = [&C_{p_1 l_1}(\tau_1), C_{p_1 l_2}(\tau_1), \cdots, C_{p_{ms} l_{ms}}(\tau_1), \cdots, C_{p_1 l_1}(\tau_2), C_{p_1 l_2}(\tau_2), \cdots \\ &, C_{p_{ms} l_{ms}}(\tau_2), \cdots, C_{p_1 l_1}(\tau_{nt}), C_{p_1 l_2}(\tau_{nt}), \cdots, C_{p_{ms} l_{ms}}(\tau_{nt})]^{\mathrm{T}} \end{aligned} \tag{4}$$

The above index vector can also be applied to single type of sensors which is a special case for the CBMSDD vector.

Definition of Damage Model

Let $\Delta\boldsymbol{\alpha}$ denote the vector of fractional change in the stiffness and mass of the elements. A linear damage model is adopted in this study .To keep the structural connectivity, the damaged system stiffness matrix $\mathbf{K_d}$ and mass matrix $\mathbf{M_d}$ are expressed as:

$$\begin{cases} \mathbf{K_d} = \sum_{i=1}^{ne} (1 + \Delta\alpha_i) \cdot \mathbf{K}_i \\ \mathbf{M_d} = \sum_{i=1}^{ne} (1 + \Delta\alpha_i) \cdot \mathbf{M}_i \end{cases} \tag{5}$$

Where $\Delta\alpha_i$ is the fractional change of both stiffness and mass in the i^{th} element; ne is the total number of finite elements in the structure.

Algorithm of Damage Detection

The sensitivity matrix $\mathbf{S}$, i.e. the sensitivity of the CBMS vector to the stiffness and mass factional change vector, can be written in the following and computed with the finite difference method.

$$\mathbf{S} = \lim_{\Delta\boldsymbol{\alpha}\to 0} \frac{\mathbf{V}_{pl}{}^{c}(\boldsymbol{\alpha}+\Delta\boldsymbol{\alpha}) - \mathbf{V}_{pl}{}^{c}(\boldsymbol{\alpha})}{\Delta\boldsymbol{\alpha}} = \left[\frac{\partial \mathbf{V}_{pl}{}^{c}}{\partial \alpha_1}, \frac{\partial \mathbf{V}_{pl}{}^{c}}{\partial \alpha_2} \cdots \frac{\partial \mathbf{V}_{pl}{}^{c}}{\partial \alpha_{ne}}\right] \tag{6}$$

where the superscript c means the CBMS vector calculated from the finite element model. The damage identification equation can then be expressed as the first-order Taylor Expansion as follows:

$$\mathbf{S}\cdot\Delta\boldsymbol{\alpha} = \Delta\mathbf{V}_{pl} = \mathbf{V}_{pl}{}^{m} - \mathbf{V}_{pl}{}^{c} \tag{7}$$

where the superscript m means the CBMS vector obtained from measured structural responses. The iterative Gaussian-Newton method is used to solve the damage identification equation.

$$\mathbf{S}^{k}\cdot\Delta\boldsymbol{\alpha}^{k+1} = \Delta\mathbf{V}_{pl}{}^{k} \quad (k = 0,1,2\cdots) \tag{8}$$

Regularization Solution with Two Objective Functions for Damage Detection

One of the effective regularization methods for damage detection is the adaptive Tikhonov regularization (Li and Law 2010). The adaptive Tikhonov regularization employs a squared norm (L_2-norm) optimization objective function, but this objective function cannot produce a sparsity solution to shrink the damage candidate locations. Zou and Hastie (2005) proposed an elastic net method including a L_1-norm penalizing term to produce a sparse solution. The elastic net method can give a sparsity result of shrinkage of the unknown identification variables into a smaller subset in terms of potential damage locations without iteration, whereas the adaptive Tikhonov regularization method can conduct an iterative improvement for identified damage severities within the subset of potential damage locations. In order to combine the advantage of the two regularization methods, the two types of objective functions mentioned above will be used successively in this study and can be expressed by Eq. (9) in terms of a tuning parameter $\beta \in [0,1]$.

$$\begin{aligned} J(\Delta\boldsymbol{\alpha},\lambda,\beta) &= \arg\min_{\Delta\boldsymbol{\alpha}}\left[\left\|\mathbf{S}\cdot\Delta\boldsymbol{\alpha} - \Delta\mathbf{V}_{pl}\right\|_2^2 + \lambda\cdot \mathrm{P}_{\beta}(\Delta\boldsymbol{\alpha})\right] \\ &= \begin{cases} \arg\min\limits_{\Delta\boldsymbol{\alpha}}\left[\left\|\mathbf{S}\cdot\Delta\boldsymbol{\alpha} - \Delta\mathbf{V}_{pl}\right\|_2^2 + \lambda\cdot\left((1-\beta)\cdot\left\|\Delta\boldsymbol{\alpha}\right\|_2^2 + \beta\cdot\left\|\Delta\boldsymbol{\alpha}\right\|_1\right)\right]; & if\ (0.0 < \beta \le 1.0) \\ \arg\min\limits_{\Delta\boldsymbol{\alpha}^{k+1}}\left[\|\mathbf{S}^{k}\cdot\Delta\boldsymbol{\alpha}^{k+1} - \Delta\mathbf{V}_{pl}{}^{k}\|_2^2 + \lambda\cdot\|\Delta\boldsymbol{\alpha}^{k+1} + \sum_{r=1}^{k}\Delta\boldsymbol{\alpha}^{r} - \boldsymbol{\alpha}^{k,*}\|_2^2\right]; & if\ (\beta = 0) \end{cases} \end{aligned} \tag{9}$$

where $\mathrm{P}_{\beta}(\Delta\boldsymbol{\alpha})$ is a penalty term; λ is the regularization parameter that governs the contribution of the two errors between $\left\|\mathbf{S}\cdot\Delta\boldsymbol{\alpha} - \Delta\mathbf{V}_{pl}\right\|_2^2$ and $\mathrm{P}_{\beta}(\Delta\boldsymbol{\alpha})$; $\left\|\Delta\boldsymbol{\alpha}\right\|_1$ is the sparsity-inducing term (Donoho 2006; Zou and Hastie 2005); the superscript k denotes the current iteration number and the term $\alpha^{k,*}$ is an adaptive adjustment factor (Li and Law 2010). How to use the two objective functions in the two-stage covariance-based multi-sensing damage detection method will be discussed in the next section.

NUMERICAL STUDY

Description of the Structure

An overhanging steel beam is employed in the experimental study to examine the feasibility and accuracy of the proposed new framework for damage detection. The FE model of the beam consists of 41 nodes and 40 equal-length beam elements. The 22nd element is cut with both 15% stiffness and mass reduction as shown in Figure 1. The acceleration and strain responses are used in this study to assemble the normalized multi-sensing data set for the subsequent study.

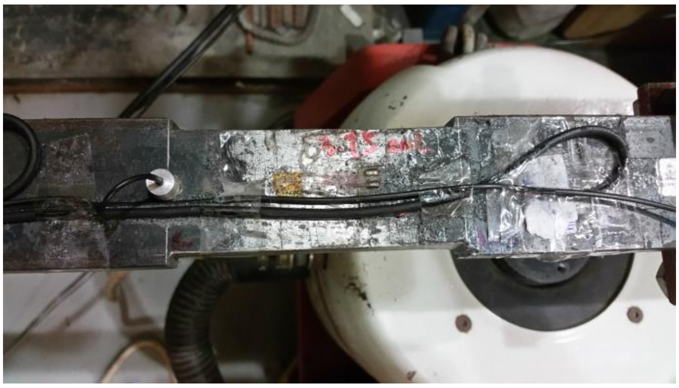


Figure 1. Damage in the steel beam

Two-stage Damage Detection

Many existing damage detection methods can find satisfactory results under the condition that the measurement noise is not considered or sufficient number of sensors are used. However, the measurement noise and limited number of sensor are inevitable in practice. Consider this two problems, this paper proposes an anti-noise damage detection index CBMS vector and the two-stage identification approach which will facilitate the damage detection under the case with limited number of sensor. Therefore the subsequent study will validate these effects.

With the aim of maximizing the data information so that structural dynamic behavior can be fully characterized, the effective independence (EfI) (Kammer 1991) technique is employed for optimal sensor placement in this study. The first four modes make major contributions to the dynamic responses. Therefore, at least four accelerometer are needed and selected as shown in Figure 2.

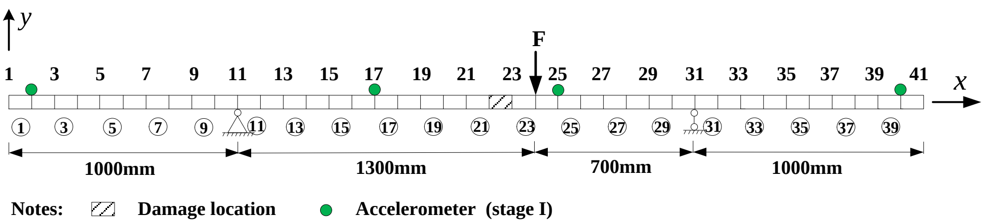
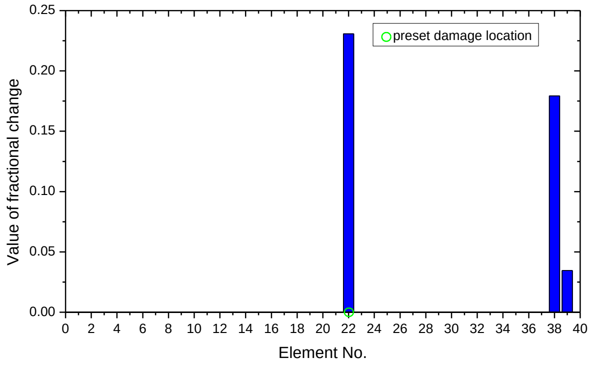


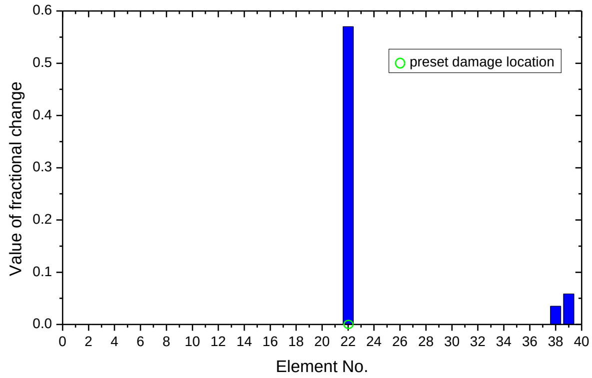
Figure 2. The initial optimal sensor configuration

It is assumed that the damage only will reduce the structure stiffness and mass. Thus, the damage inducing fractional changes are restrained in the area of [0, 1]. The fractional change value equal to 0 standing for no damage, and the fractional change value equal to 1 which hints totally damage. Under the consideration that there are usually only few damages in the structure, thus the sparse approach using the elastic net method is suitable for fast identification of initial damage candidate locations. The initial location identification result is show in the Figure.3 (a). It is noted that the element 22, 38 and 39 are identified as the initial damage location candidates. The preset damage location in element 22 is accurately identified, but there is a large false error in the element 38. Although the elastic net method can efficiently offer a sparse solution without iteration, the weak point is that it will be difficult to distinguish the real damage locations once the large false alarm with similar fractional change value as the one occurs in the element 38. This large false alarms are caused by some residual of model error and the measurement uncertainty. In order to solve this problem, the adaptive Tikhonov regularization with mature and stable iteration algorithm is used in the further refinement. The refinement location identification is now iteratively improved among the small subset of element 22, 38 and 39. The refinement result is shown in Figure.3

(b). It is noted that the fractional change value in element 22 takes most obviously change comparing with those in element 38 and element 39. This is the stage one and it hints that the element 22 is the most possible damage location.



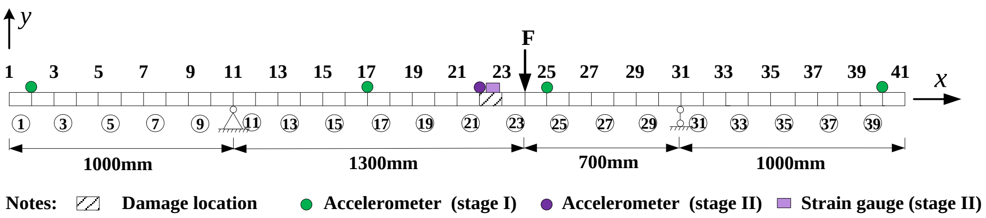
(a) Damage candidate locations shrinkage using elastic net method



(b) Damage location refinement using adaptive Tikhonov regularization

Figure 3. Damage location identification in stage one

After the stage one, the most possible damage candidate is found and the fraction change values are consider as the initial damage extent. The second stage is arranged for damage confirmation and identification improvement. This study will compare that effect by using one additional accelerometer or one additional strain gauge to enhance the local damage information, and sensor configuration is shown in Figure.4. The damage identification refinement result is shown in Figure.5. When add one accelerometer close to the damage location, the damage extent has some improvement, but the value is still large than the preset value. When add one strain gauge on the damage element, the damage extent is nearly perfectly identified. This hints that the strain gauge is more sensitive to the local damage comparing with the accelerometer. It is noted that the case using 4 accelerometer and 1 strain gauge is the multi-sensing approach and it has some advantage to give more accurate damage detection results. It needs to point out that this two stage damage detection approach has a disadvantage which is that it is executed in two stages and the second stage needs the second time sensor installation. Optimal placement for different kinds of sensors in one time and response estimation technique for unmeasured response instead of adding more sensors are the research work needing further study to improve the proposed two-stage damage detection method in the future.



Fgiure 4 The sensor configuration in the stage two

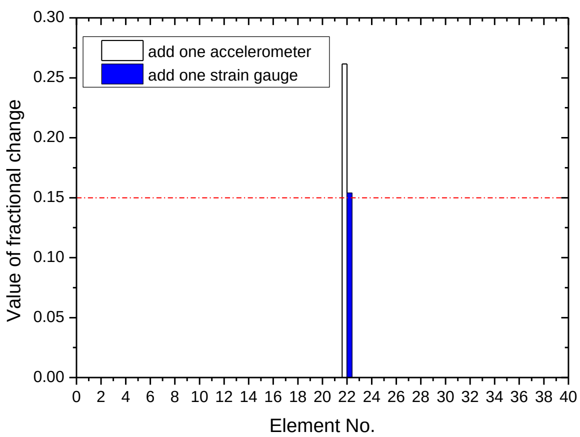


Figure 5. Damage identification refinement in stage two

CONCLUSIONS

A two-stage covariance-based multi-sensing (CBMS) damage detection method has been presented and validated with experiment in this study. Instead of using the heterogeneous measurement data separately, the new framework can assimilate and normalize the heterogeneous data simultaneously, define the CBMS vector as a new damage index in terms of the normalized cross-covariance matrix, and work together with the sensitivity approach for damage detection. It can come to the conclusions that the CBMS vector is relatively insensitive to the measurement noise, but sensitive to damage and that the dual-type sensor configuration is better for damage detection with higher accuracy. This experiment tests executed to validate the multi-sensing damage detection approach are effective, but one of unsolved disadvantage is that the dual type sensor cannot place together in the beginning and second time sensor installation is not practical to some extent.

ACKNOWLEDGMENTS

The authors wish to acknowledge the financial supports from the Research Grants Council of the Hong Kong (PolyU 5289/12E). Any opinions and conclusions presented in this paper are entirely those of the authors.

REFERENCES

Bendat, J. S., and Piersol, A. G. (1993). *Engineering Applications of Correlation and Spectral Analysis*, Wiley, New York.

Chan, W. S., Xu, Y. L., Ding, X. L., and Dai, W. J. (2006). "An integrated GPS–accelerometer data processing technique for structural deformation monitoring." *Journal of Geodesy*, 80(12), 705-719.

Donoho, D. L. (2006). "For most large underdetermined systems of linear equations the minimal." *Communications on pure and applied mathematics*, 59(6), 797-829.

Kammer, D. C. (1991). "A Hybrid Approach to Test-Analysis-Model Development for Large Space Structures." *Journal of Vibration and Acoustics-Transactions of the Asme*, 113(3), 325-332.

Law, S. S., Li, X. Y., Zhu, X. Q., and Chan, S. L. (2005). "Structural damage detection from wavelet packet sensitivity." *Engineering Structures*, 27(9), 1339-1348.

Lee, E. T., Rahmatalla, S., and Eun, H. C. (2013). "Damage detection by mixed measurements using accelerometers and strain gages." *Smart Materials and Structures*, 22(7), 075014.

Li, X. Y., and Law, S. S. (2010). "Adaptive Tikhonov regularization for damage detection based on nonlinear model updating." *Mechanical Systems and Signal Processing*, 24(6), 1646-1664.

Sim, S. H., Spencer, B. F., and Nagayama, T. (2011). "Multimetric sensing for structural damage detection." *Journal of Engineering Mechanics*.

Studer, M., and Peters, K. (2004). "Multi-scale sensing for damage identification." *Smart Materials and Structures*, 13(2), 283-294.

Sung, S. H., Park, J. W., Nagayama, T., and Jung, H. J. (2014). "A multi-scale sensing and diagnosis system combining accelerometers and gyroscopes for bridge health monitoring." *Smart Materials and Structures*, 23(1).

Zhang, X. H., Zhu, S., Xu, Y. L., and Homg, X. J. (2011). "Integrated optimal placement of displacement transducers and strain gauges for better estimation of structural response." *International Journal of Structural Stability and Dynamics*, 11(03), 581-602.

Zou, H., and Hastie, T. (2005). "Regularization and variable selection via the elastic net." *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 67(2), 301-320.